PHYSICAL REVIEW E VOLUME 56, NUMBER 5 NOVEMBER 1997

Regularity and reversibility of cascading systems

Scott Pratt and Eric Eslinger
Department of Physics and Astronomy and National Superconducting Cyclotron Laboratory, Michigan State University, East Lansing, Michigan 48824-1321
(Received 18 February 1997)

The cascading of self-organized systems such as word processor line wraps or sandpile avalanches proceeds through complex dynamics maintaining the average balance through the gradual buildup and sudden release of stress through avalanches. Using simple algebraic arguments, we argue that the distribution of time differences between successive avalanches depends crucially on the reversibility of the system. For instance, in reversible systems avalanches never occur one immediately after the other, while in irreversible systems, successive avalanches are less anticorrelated. These arguments are confirmed by line-wrap and sandpile simulations of both reversible and irreversible systems. [S1063-651X(97)08611-X]

PACS number(s): 05.40.+j, 07.05.Tp, 89.80.+h, 91.30.Px

Sandpiles, avalanches, and the line-wrap feature in a word processor are all examples of cascading systems that are self-organized. The term ''self-organized'' is inspired by the fact that the system maintains its average conditions without external input, through the gradual buildup and sudden release of stress. Power-law behavior, or correlations on all length scales, has been observed in many of these models, which has inspired them to be labeled as self-ordered critical (SOC) systems [1,2]. Attempts have been made to understand such behavior in the context of an assortment of numerical and experimental examples, such as interface growth [3], sandpiles [4], numerical evolution models [5,6], earthquakes, and sliding blocks [7].

Here we discuss basic properties and the behavior of catastrophic cascades, those cascades that release stress such as the avalanche of snow on a mountainside. We show that in reversible systems a second catastrophe cannot immediately succeed a previous catastrophe, whereas for an irreversible process, the probability of a second catastrophe is nonzero, even immediately after the previous one. Reversible systems are nondissipative by definition. Although the reversible systems that we will consider do cascade according to power-law distributions, they can reach their equilibrium state instantly, unlike dissipative systems, which must approach the equilibrium (critical) state gradually. In fact, some authors require dissipation for a system to acquire the SOC label [8]. The definition of a critical state and the approach to one is well described for a variety of numerical models in a recent study by Paczuski, Maslov, and Bak [9]. Since the purpose of this study is to understand the effects of irreversibility, we emphasize that we are not confining ourselves to SOC systems, but to any cascading systems that are self-organized.

Insight into the importance of reversibility can be gained by considering the general expression for the probability of a catastrophe:

$$\frac{d\Omega}{dn}=-\Gamma(n)\Omega, \tag{1}$$

where n is the stress added since the previous catastrophe and Ω is the probability of not having yet had a catastrophe. Thus $\Omega(n=0)=1$ and $-d\Omega/dn$ is the probability for the catastrophe to occur at the addition of the nth element of stress. If there were no correlations related to the previous state of the system, $\Gamma(n)$ would be constant and the survival probability would be a simple exponential $\Omega(n)=\exp(-\Gamma n)$.

For a reversible system, when the stress is reduced (e.g., removing sand grains from the top of a sandpile), the system reproduces its previous states (e.g., sand ascends the sandpile). Obviously, dissipative systems such as sandpiles are not reversible, but the line-wrap feature of a word processor [10] is such an example. If one enters words into the beginning of a long paragraph, line wraps ensue, which may propagate all the way to the end of the paragraph. By erasing words from the beginning, the resulting line wraps exactly reproduce previous states of the paragraph. Thus expressions representing the behavior of a reversible system should reflect the symmetry of exchanging n with $-n$. Since the left-hand side of Eq. (1) is odd in n, the right-hand side must be odd as well, requiring that

$$\Gamma(n)=-\Gamma(-n). \tag{2}$$

The most obvious consequence of this requirement is that the probability of a catastrophe immediately following the previous one is zero.

In the left-hand side of Fig. 1 we show the distribution of catastrophic line wraps binned by the number of characters n entered since the previous catastrophe. To generate this figure, a long paragraph was simulated using an algorithm described previously [10], where words were added to the beginning of the paragraph and a line-wrap cascade that exceeded 1000 lines was defined as a catastrophe. In this simulation the word-length distribution is assumed to be random between 1 and 12 characters, although such details do not affect the structure of the result on a scale larger than the average word length. We also assumed an infinite line length, which for practical purposes only means that a single line should hold many more characters than the average avalanche size. (Since only the number of spaces at the end of a line needs to be considered for wrapping, this prevented an extra length scale from entering the problem. Assuming a finite line length would cut off the tail of the distribution, as in that case avalanches could not be separated by much more

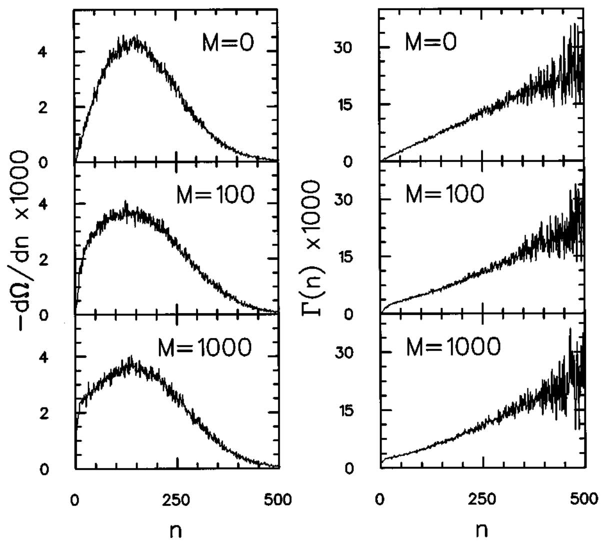

FIG. 1. Distributions of line-wrap catastrophes (cascades greater than 1000 lines) as a function of n, the number of characters entered since the preceding catastrophe, are shown in the left-hand side panels. The right-hand panels illustrate the conditional probability of having a catastrophe at n given survival until n. After each entered word M pairs of lines were switched, destroying the reversibility of the cascade. For large mixing M, the distribution is nonzero for small n and terms, even in n, appear in the ratios plotted in the right-hand-side panels.

than L characters, where L is the number of characters in a single line.) This distribution represents $-d\Omega/dn$ in Eq. (1). If one divides the distribution by the probability of surviving n characters without a catastrophe, one obtains $\Gamma(n)$, the result shown in the right-hand side of Fig. 1. This can be considered as the conditional probability of a catastrophe at the entering of the nth character given that one has survived until n without a catastrophe.

By inspecting the upper panels of Fig. 1 and by considering Eq. (1), one concludes that the conditional probability has a simple behavior $\Gamma(n)=\alpha n$. The fact that $\Gamma(n=0)$ is zero was expected from the considerations of the reversibility above, but the simplicity of the n dependence was surprising. Despite the simple behavior illustrated in the upper panels of Fig. 1, the underlying dynamics of the buildup and release of stress are complicated. Figure 2 shows the average stress (the number of characters in a line above the average) as a function of the line number for given ranges of n, the number of characters entered since the last catastrophe. The stress is concentrated at the beginning of the file in a nontrivial manner. But despite the apparent complexity of the buildup and release mechanism, the behavior illustrated in the upper panels of Fig. 1 is simple.

To demonstrate the consequences of reversibility we consider a modification to the line-wrap simulation above. Obviously, the fact that stress is building up should explain that the conditional probability illustrated in Fig. 1 is monotonically increasing, but should not explain the fact that it starts at zero. We therefore modify the simulation in such a way that the reversibility is destroyed while maintaining the buildup and conservation of stress. The modification entails performing M pairwise exchanges of the 1000 lines, which

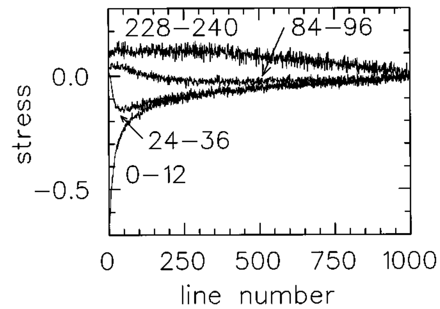

FIG. 2. For a given range of n, the characters added to the paragraph since the last catastrophe, the average stress per line is plotted. Immediately after the previous catastrophe, the stress is negative and focused at the beginning of the paragraph, while long afterward, the stress is positive and spread more evenly through the paragraph.

are performed after each word is added to the paragraph. Figure 1 demonstrates the conditional probability for three cases, $M=0$, 100, and 1000. Indeed, the conditional probability at $n=0$ moves away from zero for nonzero M and saturates for large M. Inspection of the right-hand side of Fig. 1 shows that the destruction of the reversibility introduces even terms into the expression for Γ in Eq. (1).

In the limit of large mixing M, one can solve for the conditional probability as a function of the net stress in the system by modeling the behavior with the diffusion equation, which is described by random walks at large scales. We consider a cascade where x_i characters are pushed from the $(i-1)$th line into the ith line and then $x_i+\delta_i$ characters are pushed into the $(i+1)$th line. Here δ_i can be considered a random step related to the word-length distribution. When large numbers of characters cascade through the paragraph, the variables x_i and i can be replaced with continuous variables $x(t)$ and t. Letting $f(x,t)$ refer to the probability of x characters falling into line t for a given cascade, one may describe the behavior of f with the diffusion equation

$$\frac{\partial f(x,t)}{\partial t}=D\frac{\partial^2 f(x,t)}{\partial x^2}+\beta\frac{\partial}{\partial x}f(x,t). \qquad (3)$$

Here D is the diffusion constant, which is one-half the variance of the word-length distribution [10]. The factor β represents the average stress per line $\beta\equiv s/t_c$, where t_c is the number of lines defining a catastrophic cascade and the stress s is the number of characters in the lines before t_c minus the average number of characters. If many characters have been entered since the last catastrophic cascade, the stress is positive and the distribution tends to drift to larger values of x. Immediately after a catastrophic cascade, the stress tends to be negative and the distribution drifts towards small x. There is no reason that β should not depend on t as well as the number of characters. Given the boundary condition that f approaches zero as x goes to zero, the form for f at large t is

$$f(x,t)=\frac{x}{2D}\sqrt{\frac{t_0}{t}}\exp-\frac{(x-\beta t)^2}{4Dt}. \qquad (4)$$

For the case where the stress is randomly distributed throughout the paragraph, the probability of a catastrophic cascade is

$$P_c(s)=\int_0^\infty dx\ f(x,t_c)$$
$$=\sqrt{\frac{t_0}{t_c}}\left\{e^{-\beta^2 t_c/4D}+|\beta|\sqrt{\frac{\pi t_c}{4D}}\right.$$
$$\left.\times\left[2\theta(\beta)-\mathrm{erfc}\left(|\beta|\sqrt{\frac{t_c}{4D}}\right)\right]\right\}.$$

For zero stress, the probability becomes

$$P_c(s=0)=\sqrt{\frac{t_0}{t_c}}, \tag{5}$$

while for large stress $\beta\sqrt{t_c/4D}\gg 1$,

$$P_c(s)\approx\beta\sqrt{\frac{2\pi t_0}{D}}. \tag{6}$$

Thus, for large s, the chance of a large cascade increases linearly in s and the conditional probability $\Gamma(n)$ shown in the right-hand-side figures should increase linearly with large n.

We now consider two more systems: Dhar's simple directed-sandpile algorithm [11,12] where sand falls in only two directions and a two-dimensional sandpile simulation where grains can fall in all four directions. Dhar's model is reversible, while the latter example is not.

In Dhar's algorithm, one considers a two-dimensional grid in the x and y directions, where the height z is specified at integral values of x and y. The relative height z can be either zero or one. If the height reaches 2, the height is lowered by 2 and the height of the two adjacent cells is increased by one, that is, if

$$z(x,y)\geq 2,$$

then

$$z(x,y)\rightarrow z(x,y)-2,$$
$$z(x,y+1)\rightarrow z(x,y+1)+1, \tag{7}$$
$$z(x+1,y)\rightarrow z(x+1,y)+1.$$

Thus, in this model cascades only move in two directions. Dhar has shown that this model, like the line-wrap simulation, is completely deterministic as the dynamics do not require any random choice, such as in deciding the ordering of cascades. For our purposes, we consider an array of finite width $0\leq x,y<4000$. A catastrophe is defined as a cascade that reaches the boundary $x+y=4000$. One can see that the cascade is reversible by considering what would happen if grains were subtracted from the array. By requiring that if the height goes below zero two grains are added to the height of the cell while subtracting a grain from both of the cells at higher y, one can reverse any cascade and reproduce previous states of the system.

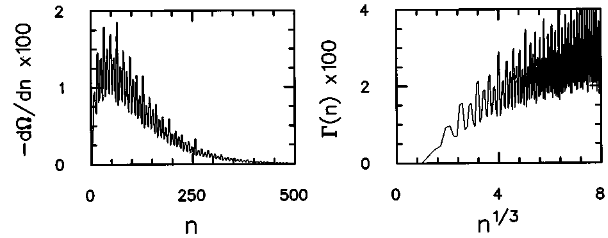

FIG. 3. Distributions of catastrophes from Dhar's directed sandpile model, binned as a function of n, the number of grains added to the top level of the pile, are shown in the left-hand-side panel. The right-hand panel illustrates the conditional probability of having a catastrophe at n given that one has survived until n. The zero intercept of the figures are in line with what one expects for reversible systems.

The pile is initialized by adding 10^6 grains randomly throughout the pile. Grains are then added to a single point $x=y=0$ and distributions are calculated for a large number of catastrophic cascades. Since the sequence of cascades can begin repeating themselves, the process is repeated many times with different initializations. Results are shown in Fig. 3. The conditional probability indeed approaches zero as expected for a reversible simulation. This algorithm induces some microscopic structure, which is ignored in the differential considerations of Eq. (1). For instance, all cascades are separated by even numbers of sand grains. Even though the distributions were binned in groups of four grains, some oscillations are still visible in the distributions.

Dhar's algorithm is no longer reversible if grains are added randomly throughout the pile. Even if the grains are only randomly added to the top three cells, $x,y\leq 1$, the reversibility is destroyed. In this case the system is a function of not just n, but also the history of where the grains were added. In that case the arguments using Eq. (1) are no longer valid and the behavior of Dhar's model becomes qualitatively like that in the more realistic sandpile discussed below.

Our final simulation has more in common with a real sandpile. Here grains are added to the center of a square table and a catastrophe is defined as an event where an avalanche removes sand from the table. We simulate the sandpile by specifying the number of grains stacked at each of 101 $\times 101$ points on a square grid. Grains are added one by one to the center of the pile. After a grain is added, the four boundaries are added to the potential-toppling list. The ordering of the directions is chosen randomly. The first toppling in the potential-toppling list is then considered. If the relative height Δh between two adjacent squares is greater than or equal to 2, a toppling may occur, which will move from zero to $\Delta h/2$ grains to the shorter stack. After a toppling, possible topplings into the hole vacated by the recent collapse are added to the potential-toppling list. Topplings of the stack that received the grains from the recent collapse into its adjacent neighbors are also added to the end of the potential-toppling list. The avalanche is not finished until all potential topplings have been considered and another grain is added to the center of the pile. This algorithm is similar to but more random in nature than the Manna model [13].

The randomness of the algorithm leads to a wide variety

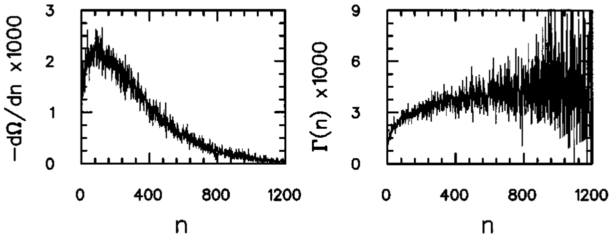

FIG. 4. Distributions of sandpile catastrophes (avalanches where sand fell off the table) as a function of n, the pile since the last catastrophe, is shown in the left-hand-side panel. The right-hand panel illustrates the conditional probability of having a catastrophe at n given that one has survived until n. The nonzero intercept of the figures is in line with what one expects for irreversible systems.

of cascades. Catastrophes may result in removing only a few grains from the table or in some cases may remove several thousand grains. The cascades may spread out and involve a significant fraction of the pile. The behavior is similar to that of the line-wrap simulation in that stress is built up incrementally and released in avalanches that are sometimes dramatic. The distribution of incremental sand added between catastrophes is shown in Fig. 4, and appears similar to the results of the line-wrap model with mixing, shown in Fig. 1. Since the sandpile model is clearly irreversible, the nonzero intercept of Fig. 4 affirms the conjecture about irreversible systems.

Here we have studied general principles of the regularity of cascading systems, where stress is built up gradually and released suddenly. We have found that reversibility plays a fundamental role in determining the system's properties through the symmetry in Eq. (2) which can be thought of as a time-reversal symmetry. In reversible systems, large avalanches cannot follow one immediately after another, while in irreversible systems, the probability of a second strong avalanche is lessened in the period immediately after the first avalanche, but it is not absolutely precluded. Since all physical systems, such as sandpiles and earthquakes, are dissipative, one should not expect the degree of regularity in the cascades as one obtains with the line-wrap simulation or with Dhar's directed sandpile algorithm. These reversible models, which have diffusive behavior as evidenced by the comparison with random walks, are nonetheless reversible and without dissipation.

Perhaps the most outstanding question regarding the regularity and predictability of avalanches involves precursors and aftershocks. In particular, what are the necessary elements for a system, either real or simulated, to have an enhanced probability of a large avalanche immediately after a previous one? If such questions can be answered, one might then classify the plethora of models and systems according to a few fundamental rules determined by their microscopic symmetries and laws of motion. This would add to one's ability to classify models through their critical exponents [14]. Since the behavior of the avalanche correlations depends smoothly on the dissipation added to the system as shown with the line-wrap model in Fig. 1, it gives hope that one might quantitatively understand the nature of correlations in successive avalanches for complex systems in terms of a few parameters, one of which surely would be a measure of dissipation.

This work was supported by the National Science Foundation Research Experience for Undergraduates Program at Michigan State, NSF Grants Nos. PHY-9424140 and PHY-9513900.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[2] P. Bak and K. Chen, Sci. Am. **264** (1), 46 (1991).
[3] K. Sneppen, Phys. Rev. Lett. **24**, 3539 (1992).
[4] J. Rosendahl, M. Vekic, and J. E. Rutledge, Phys. Rev. Lett. **73**, 537 (1994).
[5] H. Flyvbjerg, K. Sneppen, and P. Bak, Phys. Rev. Lett. **71**, 4087 (1993).
[6] C. Adami, Phys. Lett. A **203**, 23 (1995).
[7] J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989).
[8] P. Bak and M. Creutz, in *Fractals in Science*, edited by A. Bunde and S. Havlin (Springer-Verlag, Berlin, 1994), p. 26.
[9] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).
[10] W. Bauer and S. Pratt, Phys. Rev. E **54**, R1009 (1996).
[11] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).
[12] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[13] S. Manna, J. Phys. (France) **51**, 1077 (1990).
[14] A. Ben-Hur and O. Biham, Phys. Rev. E **53**, R1317 (1996).